HIGHER-DIMENSIONAL ANALOGUES OF STABLE CURVES

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ABSTRACT. The Minimal Model Program offers natural higher-dimensional analogues of stable n-pointed curves and maps: stable pairs consisting of a projective variety X of dimension ≥ 2 and a divisor B, that should satisfy a few simple conditions, and stable maps $f : (X, B) \to Y$. Although MMP remains conjectural in higher dimensions, in several important situations the moduli spaces of stable pairs, generalizing those of Deligne-Mumford, Knudsen and Kontsevich, can be constructed more directly, and in considerable generality. We review these constructions, with particular attention paid to varieties with group action, and list some open problems.

0. INTRODUCTION

Stable curves were introduced by Deligne and Mumford in [19] and proved to be extremely useful, with diverse applications in many fields of mathematics and in physics. Stable maps from n-pointed curves to varieties were used by Kontsevich to define Gromov-Witten invariants. The study of the moduli spaces of stable curves and maps is a thriving field.

Stable surfaces, the two-dimensional analogues of stable curves, were introduced by Kollár and Shepherd-Barron in [38]. It was consequently realized [4, 3] that this definition can be extended to higher-dimensional varieties and, moreover, to pairs (X, B), consisting of a projective variety X of dimension ≥ 2 and a divisor B, and to stable maps $f : (X, B) \to Y$. One arrives at this definition by mimicking the construction of stable one-parameter limits of curves in the higher-dimensional case, and replacing contractions of (-1)- and (-2)-curves by the methods of the Minimal Model Program.

Stable pairs provide an apparently very general, nearly universal way to compactify moduli spaces of smooth or mildly singular varieties and pairs. There are, however, two complications. First, as of this writing, the Minimal Model Program in arbitrary dimensions is still conjectural. Secondly, even in the case of surfaces the resulting moduli spaces turn out to be very complicated, and numerical computations similar to the curve case seem to be out of reach.

The situation can be improved in both respects by looking at some particularly nice classes of varieties, such as abelian varieties and other varieties with group action: toric, spherical, and also at varieties and pairs closely related to them, for example the hyperplane arrangements.

In all of these cases the Minimal Model Program can be used for guessing the correct answer, but the actual constructions of the moduli spaces can be made

without it, by exploiting symmetries of the situation. At the same time, the resulting moduli spaces come equipped with rich combinatorial structures, typically with stratifications labeled by various polytopal tilings.

The aim of this paper is to review the basic constructions and several of the examples mentioned above. My understanding of the subject was shaped over the years by discussions with (in the chronological order) J. Kollár, S. Mori, I. Nakamura, K. Hulek, Ch. Birkenhake, M. Brion, B. Hassett, A. Knutson, and many other people whom I am unable to list here. I am indebted to them all.

1. Definition of stable pairs and maps

To define varieties and pairs, we work over an algebraically closed field k of arbitrary characteristic. All **varieties** will be assumed to be connected and reduced but not necessarily irreducible. A **polarized variety** is a projective variety X with an ample invertible sheaf L.

Definition 1.1. Let X be a projective variety, B_j, $i = 1, \ldots, n$, be effective Weil divisors on X, possibly reducible, and b_j be some rational numbers with $0 < b_j \leq 1$. The pair $(X, B = \sum b_j B_j)$ (resp. a map $f : (X, B) \to Y$) is called **stable** if the following two conditions are satisfied:

(1) *on singularities:* the pair (X, B) is semi log canonical, and
(2) *numerical:* the divisor $K_X + B$ is ample (resp. f-ample).

Both parts require an explanation.

Definition 1.2. Assume that X is a *normal* variety. Then X has a canonical Weil divisor K_X defined up to linear equivalence. The pair (X, B) is called **log canonical** if

(1) $K_X + B$ is $\mathbb{Q}$-Cartier, i.e. some positive multiple is a Cartier divisor, and
(2) for every proper birational morphism $\pi : X' \to X$ with normal X', in the natural formula

$$K_{X'} + \pi_*^{-1} B = \pi^*(K_X + B) + \sum a_i E_i$$

one has $a_i \geq -1$. Here, E_i are the irreducible exceptional divisors of π, and the pullback π^* is defined by extending $\mathbb{Q}$-linearly the pullback on Cartier divisors. $\pi_*^{-1} B$ is the strict preimage of B.

If char $k = 0$ then X has a resolution of singularities $\pi : X' \to X$ such that $\mathrm{Supp}(\pi_*^{-1} B) \cup E_i$ is a normal crossing divisor; then it is sufficient to check the condition $a_i \geq -1$ for this morphism π only.

The definition for semi log canonical surface singularities X originated in [38]. The following definition, equivalent to [38] in the surface case, and extending it to higher-dimensional varieties and pairs, is from [3].

Definition 1.3. A pair (X, B) is called **semi log canonical** if

(1) X satisfies Serre's condition S2, in particular, equidimensional,
(2) X has at worst double normal crossing singularities in codimension one, and no divisor B_j contains any component of this double locus,
(3) some multiple of the Weil $\mathbb{Q}$-divisor $K_X + B$, well defined thanks to the previous condition, is $\mathbb{Q}$-Cartier, and
(4) denoting by $\nu : X^\nu \to X$ the normalization, the pair $(X^\nu, (\text{double locus}) + \nu^{-1} B)$ is log canonical.

Example 1.4. Assume that X is a curve. Then (X, B) is semi log canonical iff X is at worst nodal, B_j do not contain any nodes, and for every $P \in X$ one has $\operatorname{mult}_P B = \sum b_j \operatorname{mult}_P B_j \le 1$. A map $f : (X, B) \to Y$ is stable if, in addition to this condition on singularities, the divisor $K_X + B$ has positive degree on every irreducible component of X collapsed by f.

Hence, for $b_j = 1$, $\deg B_j = 1$, and $Y =$ a point (i.e. in the absolute case) these are precisely the Deligne-Mumford-Knudsen stable n-pointed curves [19, 33, 32]. With the same assumptions on B but Y arbitrary, these are Kontsevich's stable maps. Hassett [27] considered the absolute case with $0 < b_j \le 1$, $\deg B_j = 1$, for which he constructed a smooth Deligne-Mumford stack with a projective moduli space.

The motivation for the definition of stable pairs is that they appear as natural limits of one-parameter families of smooth varieties and pairs $(X, B) \to S$, as will be developed in Section 2. In higher dimensions, there is an additional complication: if the total divisor B is not $\mathbb{Q}$-Cartier then the central fiber B_0 may have an embedded component, so no longer be an ordinary divisor. There are several ways to fix this:

(1) Pairs with floating coefficients b_j. We will say that a pair

$$\left(X, B = \sum b_j B_j + \sum (b_k + \epsilon_k) B_k\right)$$

(resp. a map) is stable if, in addition, the divisors B_k in the second group are $\mathbb{Q}$-Cartier and for all $0 < \epsilon_k \ll 1$, the pair $(X, \sum b_j B_j + \sum (b_k + \epsilon_k) B_k)$ is stable.

(2) Pairs with coefficients b_j outside of a "bad" subset of $[0, 1]$. Again, the idea here is the same as in (1), to avoid the values b_j for which the total divisors B_j may be not $\mathbb{Q}$-Cartier.

(3) Working with subschemes $B_j \subset X$ instead of simply divisors.

(4) Working with finite morphisms $B_j \to X$, where B_j are (reduced) varieties of dimension $\dim B_j = \dim X - 1$, rather than with embedded divisors.

2. MINIMAL MODEL PROGRAM CONSTRUCTION

The true motivation for the introduction of stable pairs is that they inevitably appear as limits of one-parameter families of smooth varieties and pairs, when one tries to follow the classical construction in the case of curves. This is explained by the following statement, which however is conditional: it depends on the validity of the log Minimal Model Program in dimension $\dim X + 1$ (so, currently problematic for pairs (X, B) with $\dim X \ge 3$) and on Inversion of Adjunction in an appropriate sense, as explained in the sketch of the proof below. The argument also requires $\operatorname{char} k = 0$ or X to be a curve for the resolution of singularities and semistable reduction.

This statement appeared in [38] in the case of surfaces with $B = 0$, where it is not conjectural and in [3] in the more general case (see also [26]).

By a one-parameter family of stable maps we will understand a morphism $f : (X, B) \to Y \times S$, where $(S, 0)$ is a germ of a nonsingular curve, such that $\pi = p_2 \circ f : X \to S$ and $\pi|_{B_j} : B_j \to S$ are flat, and every geometric fiber $f_{\bar{s}} : (X_{\bar{s}}, B_{\bar{s}}) \to Y$ is a stable map. We will denote $S \setminus 0$ by U.

The definition of a family over an arbitrary scheme S is similar but requires care, especially if S is not reduced. We will discuss it in the next Section.

Theorem 2.1 (Properness of the functor of stable maps)**.** *Every punctured family* $f_U : (X_U, B_U) \to Y \times U$, *of stable pairs has at most one extension to a family of stable pairs over S. Moreover, such an extension does exist after a finite base change* $(S', 0) \to (S, 0)$.

Sketch of the proof. We assume that fibers X_s for $s \neq 0$ are normal, for simplicity. Denote an extension by $f : (X, B) \to Y \times S$. Inversion of Adjunction of Shokurov-Kollár (see, e.g. [35], 17.3) says that the central fiber (X_0, B_0) is semi log canonical iff the pair $(X, B + X_0)$ is log canonical.

Now suppose that we have an extension. Then $(X, B + X_0)$ has log canonical singularities and $K_X + B + X_0$ is f-ample. Hence, $(X, B + X_0)$ is *the log canonical model* of $(\widetilde{X}, \widetilde{B} + \widetilde{X}_{0,\mathrm{red}})$ over $Y \times S$ for any resolution of singularities $(\widetilde{X}, \widetilde{B})$ of any extension of f_U. Existence of the log canonical model is the main result of the log Minimal Model Program, and its uniqueness is a basic and easy fact, see, e.g. [39].

In the opposite direction, pick some extension family. Take a resolution of singularities, which introduces some exceptional divisors E_i. Apply the Semistable Reduction Theorem to this resolution. The result is that after a ramified base change $(S', 0) \to (S, 0)$ we now have an extended family $\tilde{f}' : (\widetilde{X}', \widetilde{B}')$ such that $\widetilde{X}'$ is smooth, the central fiber $\widetilde{X}'_0$ is a reduced normal crossing divisor, and, moreover, $\widetilde{X}'_0 \cup \mathrm{Supp}\, \widetilde{B}' \cup \widetilde{E}'_i$ is a normal crossing divisor.

It follows that the pair $(\widetilde{X}', \widetilde{B}' + \widetilde{X}'_0 + \sum \widetilde{E}'_i)$ has log canonical singularities and is relatively of general type over $Y \times S'$. Now let $f' : (X', B' + X'_0) \to Y \times S'$ be its log canonical model, guaranteed by the log Minimal Model Program. The divisor $K_{X'} + B' + X'_0$ is f'-ample. Inversion of Adjunction – applied in the opposite direction now – guarantees that the central fiber (X'_0, B'_0) has semi log canonical singularities. Finally, since (X_U, B_U) has log canonical singularities, outside the central fiber the log canonical model of $(\widetilde{X}'_U, \widetilde{B}'_U + \sum \widetilde{E}'_{i,U})$ coincides with $(X_U, B_U) \times_U U'$. So we obtained the desired extension. □

3. Surfaces

The situation with the moduli spaces of surfaces is as follows. The broad outline has been understood for a long time, see [38, 34, 4], but answers to several thorny technical questions have been published only recently. With these technical questions resolved, for any fixed projective scheme Y one can construct the moduli space of stable maps $f : (X, B) \to Y$ with B empty or reduced (i.e. with all $b_j = 1$), as a projective scheme. For the arbitrary coefficients b_j, one faces the difficulties with subschemes B_j acquiring embedded components (an example due to Hassett shows that this really happens), and the technical details of the solution are yet to be published. We now give a brief overview.

Definition of the moduli functor. We choose a triple of positive rational numbers $C = (C_1, C_2, C_3)$ and a positive integer N. We also fix a very ample sheaf $\mathcal{O}_Y(1)$ on Y. Then the basic moduli functor $M_{C,N}$ associates to every Noetherian scheme S over a base scheme the set $M_{C,N}(S)$ of maps $f : (X, B) \to Y \times S$ with the following properties:

(1) X and B_j are flat schemes over S.
(2) The double dual $\mathcal{L}_N(X/S) = \left(\omega_{X/S}^{\otimes N} \otimes \mathcal{O}_X(NB)\right)^{**}$ is an invertible sheaf on X, relatively ample over $Y \times S$.

(3) For every geometric fiber, $(K_{X_s} + B_s)^2 = C_1$, $(K_{X_s} + B_s)H_s = C_2$, and $H_s^2 = C_3$, where $\mathcal{O}_X(H) = f^*\mathcal{O}_Y(1)$.

Kollár suggested a different moduli functor, of families for which the formation of the sheaves $\mathcal{L}_N(X/S)$ commutes with arbitrary base changes $S' \to S$, i.e.

$$\mathcal{L}_N(X \times_S S'/S') = \phi^*\mathcal{L}_N(X/S)$$

for *all* sheaves $\mathcal{L}_N$ for which NB is integral (e.g., all $N \in \mathbb{Z}$ if B is reduced).

Boundedness. Boundedness means that for any stable map over an algebraically closed field, with fixed invariants C_1, C_2, C_3, there exists N such that the sheaf $L_N = \mathcal{O}_X(N(K_X + B))$ is invertible. Then it is easy to prove that for a fixed multiple M of N the sheaf L_M is very ample with trivial higher cohomologies.

For surface pairs with fixed b_j boundedness was proved in [2], see also [14] for a somewhat simpler, and effective version. For the stable maps it was proved in [4]. (We also note that Karu [30] proved boundedness for *smoothable* stable varieties of dimension d assuming Minimal Model program in dimension $d + 1$.)

Local closedness. This means that for every family of pairs $f : (X, B) \to Y \times S$, with fibers not assumed to be stable pairs, there exists a locally closed subscheme $U \to S$ with the following universal property: For every $S' \to S$, the pullback family represents an element of $M_N(S')$ if and only if $S' \to S$ factors though U. An important case of this statement from which the general case follows, was established in [28].

Construction of the moduli space. Let $f : (X, B) \to Y \times S$ be a family of stable maps over S. By boundedness, for some fixed multiple M of N, the sheaf $\pi_*\mathcal{L}_M$ is locally free, so it can be trivialized on an open cover $S = \cup S_i$. With such trivializations chosen, the graphs of the maps $f_i = f|_{S_i}$ are closed subschemes of $\mathbb{P}^n \times Y$, and so represent a collection of S_i-points of the Hilbert scheme $\text{Hilb}_{\mathbb{P}^n \times Y, p}$, for an easily computable Hilbert polynomial p. For a different choice of trivializations, the points differ by the action of $\text{PGL}_{n+1}(S_i)$.

By local closedness, there exist a locally closed subscheme $U \to \text{Hilb}_{\mathbb{P}^n \times Y, p(x)}$ such that the above S_i-points of the Hilbert scheme are S_i-points of U. Vice versa, every morphism $S \to U$ gives a family of stable maps over S.

It follows that the moduli functor is the quotient functor U/PGL_{n+1}. The separatedness of the moduli functor implies that the PGL_{n+1}-action is proper. Then the quotient exists as an algebraic space by applying either [36] or [31]. It is a proper algebraic space because the moduli functor is proper.

Projectivity of the moduli space. Kollár [34] provided a general method for proving projectivity of complete moduli spaces. It applies in this situation with minor modifications. In particular, the moduli space is a scheme.

We note that the quasiprojectivity of the open part corresponding to arbitrary-dimensional polarized varieties with canonical singularities was proved by Viehweg [48] by using methods of Geometric Invariant Theory.

A floating coefficient version. Hacking [22] constructed a moduli space for the stable pairs $(\mathbb{P}^2, (3/d + \epsilon)B)$, where B is a plane curve of degree d, and their degenerations.

Other special surfaces. Other papers treating special cases include [1, 25, 47].

4. Toric and spherical varieties

In terms of Definition 1.1, this case corresponds to the pairs $(X, \Delta + \epsilon B)$, with a floating coefficient. The following very easy statement is the main bridge connecting the log Minimal Model Program and stable pairs with the combinatorics of toric varieties.

We fix a multiplicative torus $T = \mathbb{G}_m^r$. Below, **toric variety** means a normal variety with T-action and an open T-orbit; no special point is chosen (as opposed to a torus embedding).

Lemma 4.1 ([3]). *Let X be a projective toric variety, Δ be the complement of the main T-orbit, and B be an effective $\mathbb{Q}$-Cartier divisor. Then the pair (X, Δ) has log canonical singularities, and the pair $(X, \Delta + \epsilon B)$ with effective divisor B and $0 < \epsilon \ll 1$ is stable iff B is an ample Cartier divisor which does not contain any T-orbits.*

Proof. For any toric variety one has $K_X + \Delta = 0$. In particular, we can apply this to X and to a toric resolution of singularities $\pi : \widetilde{X} \to X$. The divisor $\widetilde{\Delta}$, the union of $\pi_*^{-1}\Delta$ and the exceptional divisors E_i, is a normal crossing divisor. But then the formula $K_{\widetilde{X}} + \widetilde{\Delta} = 0 = \pi^*(K_X + \Delta)$ says that the discrepancies a_i in the formula $K_{\widetilde{X}} + \pi_*^{-1}\Delta = \pi^*(K_X + \Delta) + \sum a_i E_i$ all equal -1.

For the pair $(X, \Delta + \epsilon B)$ to be stable, B must be $\mathbb{Q}$-Cartier and ample, since $K_X + \Delta + \epsilon B = \epsilon B$. By continuity of discrepancies of $(X, \Delta + \epsilon B)$ in ϵ, we see that the latter pair is log canonical iff π^*B does not contain any irreducible components of $\widetilde{\Delta}$, i.e. the closures of proper T-orbits on $\widetilde{X}$. Equivalently, B should not contain any T-orbits. Finally, any effective Weil divisor not containing a T-orbit is Cartier. $\square$

Definition 4.2. Let X be a variety with T-action, and $B \subset X$ be an effective Cartier divisor. The variety X, resp. the pair (X, B) is called a **stable toric variety** (resp. **stable toric pair**) if the following three conditions are satisfied:

   (1) *on singularities:* X is seminormal (resp. and B does not contain any T-orbits),
   (2) *on group action:* isotropy groups T_x are subtori (so connected and reduced), and there are only finitely many orbits,
   (3) *numerical:* (resp. the divisor B is ample).

A **family of stable toric pairs** is a proper flat morphism $f : (X, B) \to S$, where X is a scheme endowed with an action of $T_S := T \times S$, with a relative Cartier divisor B, so that every geometric fiber is a stable toric pair. We will denote the invertible sheaf $\mathcal{O}_X(B)$ by L.

A polarized T-variety (X, L) is **linearized** if X is projective, and the sheaf L is provided with a T-linearization.

We see that a pair (X, B) with a toric variety X is a stable toric pair iff the pair $(X, \Delta + \epsilon B)$ is a stable pair in the sense of Definition 1.1. We also note that the boundary Δ is determined by the group action, and so can be omitted.

One proves rather easily that a linearized stable toric variety is a union of (normal) polarized toric varieties (X_i, L_i) which, as it is well known [44], correspond to lattice polytopes Q_i. In this way, one obtains a **complex of polytopes** $\mathcal{Q} = (Q_i)$, and X is glued from the varieties X_i combinatorially in the same way as the topological space $|\mathcal{Q}|$ is glued from Q_i. The complex $\mathcal{Q}$ comes with a **reference map**

$\rho : |\mathcal{Q}| \to M_{\mathbb{R}}$, where M is the character group of T, identifying each cell Q_i with a lattice polytope. The pair $(|\mathcal{Q}|, \rho)$ is called the **type** of a stable toric variety.

A section $s \in H^0(X, L)$ with $B = (s)$ gives a collection of sections

$$s_i = \sum s_{i,m} e^m \in H^0(X_i, L_i) = \oplus_{m \in Q \cap M} k e^m$$

For each polytope Q_i this gives a subset $C_i = \{m \mid s_{i,m} \neq 0\}$ and, since B does not contain any T-orbits, one must have $\operatorname{Conv} C_i = Q_i$. This defines a **complex of marked polytopes** $(\mathcal{Q}, \mathcal{C})$.

4.3. All stable toric varieties X (resp. pairs (X, B)), are classified, up to an isomorphism, by the following data:

(1) A complex of polytopes $\mathcal{Q}$ with a reference map $\rho : |\mathcal{Q}| \to M_{\mathbb{R}}$ (resp. a complex of marked polytopes $(\mathcal{Q}, \mathcal{C})$ with a reference map).

(2) An element of a certain cohomology group which we briefly describe.

For each polytope $Q_i \in \mathcal{Q}$, let $\widetilde{M}_i \subset \widetilde{M} = \mathbb{Z} \times M$ be the saturated sublattice of $\widetilde{M}$ generated by $(1, Q_i)$, and let $\widetilde{T}_i = \operatorname{Hom}(\widetilde{M}_i, \mathbb{G}_m)$ be the corresponding torus. The collection of stalks $\{\widetilde{T}_i\}$ defines the sheaf $\underline{\widetilde{T}}$ on the complex $\mathcal{Q}$. Then the set of isomorphism classes of polarized stable toric varieties is simply $H^1(\mathcal{Q}, \underline{\widetilde{T}})$, and each of them has automorphism group $H^0(\mathcal{Q}, \underline{\widetilde{T}})$.

Similarly, one defines the sheaf $\underline{\widehat{C}} = \operatorname{Hom}(\mathcal{C}, \mathbb{G}_m)$ with the stalks $\operatorname{Hom}(C_i, \mathbb{G}_m)$ in which the sections s_i live. The natural sheaf homomorphism $\underline{\widetilde{T}} \to \underline{\widehat{C}}$ gives a homomorphism of cochain complexes $\phi : C^*(\underline{\widetilde{T}}) \to C^*(\underline{\widehat{C}})$. Then the first cohomology of the cone complex $\operatorname{Cone}(\phi)$ is the set of isomorphism classes of stable toric pairs of type $(\mathcal{Q}, \mathcal{C})$, and the zero cohomology gives the automorphism groups of the pairs. These automorphism groups are finite.

The following Lemma also goes back to [3].

Lemma 4.4. *Suppose that the topological space $|\mathcal{Q}|$ is homeomorphic to a manifold with boundary. Let (X, B) be a stable toric pair in the sense of Definition 4.2. Let Δ be the reduced divisor corresponding to the boundary of $|\mathcal{Q}|$. Then the pair $(X, \Delta + \epsilon B)$ is stable in the sense of Definition 1.1.*

Proof. One proves that with the above assumption on $|\mathcal{Q}|$ the variety X is Cohen-Macaulay, a fortiori, satisfies S2, and has only simple crossings in codimension 1 (each component of the double locus corresponds to a codimension-1 polytope in $\mathcal{Q}$ which is a face of two maximal-dimensional polytopes). The normalization of $(X, \Delta + \epsilon B)$ with the double locus added is the disjoint union of toric pairs $(X_i, \Delta_i + \epsilon B_i)$, which are log canonical by Lemma 4.1. Hence, $(X, \Delta + \epsilon B)$ is semi log canonical. Moreover,

$$\nu^*(K_X + \Delta + \epsilon B) \mid_{X_i} = K_{X_i} + \Delta_i + \epsilon B_i,$$

so the divisor $K_X + \Delta + \epsilon B$ is ample. $\qquad\square$

We would like to mention the following essential facts: Higher cohomology groups of positive powers L^d vanish. The moduli functor of stable toric pairs is proper, i.e. every one-parameter family has at most one limit, and the limit always exists after a finite base change $(S', 0) \to (S, 0)$. The limit of a family of pairs of type $\mathcal{Q}$ corresponds to a complex $\mathcal{Q}'$ such that $|\mathcal{Q}'| = |\mathcal{Q}|$, and $\mathcal{Q}'$ is obtained from $\mathcal{Q}$ by a *convex* subdivision.

Recall that a subdivision of a single polytope Q is **convex** if it is the projection of the lower envelope of several points $(m, h(m))$ where m are some points with $\mathrm{Conv}(m) = Q$, and $h : \{m\} \to \mathbb{R}$ is an arbitrary function, called **height function**. This was generalized in [6] to convex subdivisions of a polytopal complex $\mathcal{Q}$ by requiring that the height functions of two polytopes Q_1, Q_2 differ by a linear function on $Q_1 \cap Q_2$.

The stable toric variety X is **multiplicity-free** if the reference map $\rho : |\mathcal{Q}| \to M_{\mathbb{R}}$ is injective. We will restrict ourselves to this case for the rest of this Section.

Theorem 4.5 ([6]). *The functor of stable toric pairs has a coarse moduli space M over $\mathbb{Z}$. It is a disjoint union of subschemes $M_{|\mathcal{Q}|}$, each of them projective. Each moduli space $M_{|\mathcal{Q}|}$ has a natural stratification with strata corresponding to subdivisions of $|\mathcal{Q}|$ into lattice polytopes.*

When $|\mathcal{Q}| = Q$ is a polytope, the moduli space M_Q contains an open subset U_Q which is the moduli space of pairs (X, B) with a toric variety X. The closure of U_Q is an irreducible component of M_Q. The strata in this closure correspond to convex subdivisions of Q, and the normalization of $\overline{U}_Q$ coincides with the toric variety for the secondary polytope of $(Q, Q \cap M)$.

Rather than relying on the methods of the Minimal Model Program, the proof is rather direct. To each family $(T_S \curvearrowright X, B) \to S$ we can associate the graded algebra $R(X/S, L) = \oplus_{d \geq 0} \pi_* L^d$, multigraded by M due to the T_S-action, and multiplicity-free by the assumptions on the fibers, with a section s, an equation of B. Then the moduli of stable toric pairs is equivalent to the moduli of algebras (R, s) with a section, and the latter is rather straightforward.

We note that the faces of the secondary polytope of $(Q, Q \cap M)$ (see [20] for the definition) are in bijection with the convex subdivisions of Q.

For some polytopes Q the moduli space M_Q does indeed have several irreducible components. The extra components always appear when there exists a non-convex subdivision of Q into lattice polytopes.

Another situation where the extra components are guaranteed is when the stratum for a particular convex subdivision has higher dimension in M_Q than it does in the toric variety for the secondary polytope. Both can be computed effectively: the latter is the codimension of the corresponding cone in the normal fan of the secondary polytope, and for M_Q it is the dimension of the cohomology group describing the gluing, as in 4.3.

On the other hand, the following dual point of view turns out to be very important.

Definition 4.6. Let Y be a projective scheme. A **variety over Y** is a reduced, but possibly reducible, projective variety X together with a finite morphism $f : X \to Y$. A **family of varieties over Y** is a finite morphism $f : X \to Y \times S$ such that $X \to S$ is flat, and every geometric fiber is a variety over Y, as above.

If G is an algebraic group acting linearly on $Y \subset \mathbb{P}^n$, then a G-variety (resp. family) over Y is a morphism $f : X \to Y$ as before which, in addition, is G-equivariant.

Lemma 4.7. *Families of stable toric pairs of type $|\mathcal{Q}|$ are in a natural bijective correspondence with families of stable toric varieties over $\mathbb{P}^n$ with the homogeneous coordinates x^m, $m \in M \cap |\mathcal{Q}|$, on which T acts with the characters m.*

Proof. Indeed, the data for both the morphism to $\mathbb{P}^n$ and the divisor B not containing any T-orbits is the same: working locally over $S = \operatorname{Spec} A$, it is a collection $(c_m \in A)$ such that $c_m(s) \neq 0$ for every *vertex* m of a polytope Q_i corresponding to the fiber X_s. □

Remark 4.8. We note that there exists another moduli space closely related to our moduli space M of stable toric varieties: it is the toric Hilbert scheme $\operatorname{Hilb}_{\mathbb{P}^n}^T$ [46, 24] parameterizing *subschemes* $X \subset \mathbb{P}^n$ corresponding to the multiplicity-free multigraded algebras. So, geometrically what we have done is the following: we replaced *closed subschemes* of Y by *reduced* varieties X with a *finite morphism to Y*.

Indeed, this is a general situation, and in [13] such a universal substitute for the Hilbert scheme is constructed in general, without the multiplicity-free assumption (or group action). Reduced varieties with a finite morphism to a scheme Y are called **branchvarieties of Y**, to contrast with subvarieties or subschemes of Y.

Over a field of characteristic zero, the above picture can be generalized to stable spherical varieties over Y. Recall that if G is a connected reductive group then a G-variety X is called **spherical** if it is normal and a Borel subgroup of G has an open orbit. One motivation for considering spherical varieties is the following important finiteness property: a G-variety is spherical iff any G-variety birationally isomorphic to it has only finitely many G-orbits.

A polarized G-linearized variety (X, L) is spherical iff it is normal and the algebra $R(X, L) = \oplus_{d \geq 0} H^0(X, L^d)$ is multiplicity free, i.e. when it is written as a direct sum over the irreducible representations $V_{d,\lambda}$ of the group $\widetilde{G} = \mathbb{G}_m \times G$, each multiplicity is 1 or 0.

One important difference between the toric and spherical cases is that the spherical varieties are *not* completely classified. For any homogeneous spherical variety G/H, its normal G-embeddings correspond to colored fans. However, the homogeneous spherical varieties G/H are currently only classified in types A and D, [41, 17].

For a polarized spherical variety (X, L) one can define its moment polytope [18] which, when working over $\mathbb{C}$, coincides with the moment polytope of X as a Hamiltonian variety. Which polytopes may appear as moment polytopes is not known. However, it is known that the set of moment polytopes contained in any bounded set is finite [11, 12]. Together with the boundedness results of [13], this implies that the set of moment polytopes of polarized stable spherical varieties (X, L) with a fixed Hilbert polynomial is finite.

Definition 4.9. A **stable spherical variety over** a G-variety $Y \subset \mathbb{P}^n$ is a G-variety over Y such that the G-module $R(X, L)$, $L = f^* \mathcal{O}_Y(1)$ is multiplicity-free,

Similarly, a **family of stable spherical varieties over Y** is a proper family of G-varieties $f : G_S \curvearrowright X \to Y \times S$ over Y such that, denoting $L = f^* \mathcal{O}_{Y \times S}(1)$, one has
$$R(X/S, L) = \oplus_{d \geq 0} \pi_*(X, L^d) = \oplus_\lambda V_{d,\lambda} \otimes F_{d,\lambda},$$
where $V_{d,\lambda}$ are the irreducible $(\mathbb{G}_m \times G)$-representations and $F_{d,\lambda}$ are locally free sheaves of rank 1 or 0.

There is an equivalent more geometric definition in terms of gluing from spherical varieties. However, as was noted above, the structure of the "building blocks", i.e. ordinary spherical varieties, is a little mysterious.

Theorem 4.10 ([12]). *The functor of stable spherical pairs over Y has a coarse moduli space M_Y over $\mathbb{Q}$. It is a disjoint union of projective schemes.*

As in the toric case, one can define a **stable spherical pair** (X, B). However, when G is not a torus, this turns out to be a very special case of stable maps. The analogue of Lemma 4.7 in the spherical case is the following:

Lemma 4.11 ([12], Prop.3.3.2). *Families of stable spherical pairs of type $|\mathcal{Q}|$ are in a natural bijective correspondence with families of stable spherical varieties over $\mathbb{P}\big(\oplus \operatorname{End}(V_m)\big)$, where m go over the weights in $|\mathcal{Q}|$.*

One important case where all stable spherical varieties are completely classified is the case of **stable reductive varieties** [9, 10]. Each polarized stable reductive variety corresponds to a complex $\mathcal{Q} = (Q_i)$ of lattice polytopes in $\Lambda_{\mathbb{R}}$, where Λ is the weight lattice of G, and the complex $\mathcal{Q}$ is required to be invariant under the action of Weyl group. Limits of one-parameter degenerations again correspond to convex subdivisions.

5. Abelian varieties

In terms of Definition 1.1, this case corresponds to the pairs $(X, \epsilon B)$, with a floating coefficient. Here, X is an abelian variety, or more accurately an abelian torsor (i.e. no origin is fixed) or a similar "stable" variety, and B is a theta divisor. But again, Minimal Model Program is not used, and the constructions and proofs of [6] are more direct, using the symmetries of the situation.

A fundamental insight from the Minimal Model Program is that we should be working with abelian torsors with divisors $(B \subset X)$ instead of abelian varieties $(0 \in X)$, because the former fit into the general settings of Definition 1.1 and the basic construction of Section 2, and the latter do not. The bridge is the following

Lemma 5.1 ([6]). *There is a natural bijective correspondence between principally polarized abelian schemes $(A, \lambda : A \to A^t) \to S$ and flat families of abelian torsors $(A \curvearrowright X, B) \to S$ such that B is an effective relative Cartier divisor defining a principal polarization on each geometric fiber.*

For example, if $C \to S$ is a smooth family of curves then $(\operatorname{Pic}^0_{C/S}, \lambda)$ is the principally polarized abelian scheme, and $(\operatorname{Pic}^0_{C/S} \curvearrowright \operatorname{Pic}^{g-1}_{C/S} \supset \Theta_{g-1})$ is the family of abelian torsors. The two families are usually not isomorphic, unless $C \to S$ has a section.

Recall that a **semiabelian variety** is a group variety G which is an extension

$$1 \to T \to G \to A \to 0$$

of an abelian variety by a multiplicative torus. Let $g = \dim G = r + a = \dim T + \dim A$. We will denote the lattice of characters of T by $M_0 \simeq \mathbb{Z}^r$ and reserve $M \simeq \mathbb{Z}^g$ for a certain lattice of which M_0 will be a quotient.

Generalizing directly Definition 4.2, we give the following:

Definition 5.2. Let X be a variety with an action of a semiabelian variety G, and $B \subset X$ be an effective Cartier divisor. The variety X, resp. the pair (X, B) is called a **stable quasiabelian variety** (resp. **stable quasiabelian pair**) if the following three conditions are satisfied:

(1) *on singularities:* X is seminormal (resp. and B does not contain any G-orbits),
(2) *on group action:* isotropy groups G_x are subtori, and
(3) *numerical:* (resp. the divisor B is ample).

A **family of stable quasiabelian pairs** is a proper flat morphism $f : (G \curvearrowright X, B) \to S$, where G is a semiabelian group scheme over S, X is a scheme endowed with an action $G \times_S X \to X$, with a relative Cartier divisor B, so that every geometric fiber is a stable quasiabelian pair.

The essential difference with the case of stable toric varieties is that the group variety G may vary, and in particular the torus part T may change its rank.

Intuitively (and this actually works when working over $\mathbb{C}$) a polarized abelian variety should be thought of as a stable toric variety for a *constant* torus that, in terms of the data 4.3, corresponds to a topological space $|\mathcal{Q}| = \mathbb{R}^g/\mathbb{Z}^g$ and an element of a cohomology group describing the gluing, as in 4.3.2; however, the Čech cohomology should be replaced by the group cohomology. A general polarized stable quasiabelian variety should be thought of as a similar quotient of a bigger stable toric variety for a constant torus by a lattice.

Remark 5.3. Namikawa [43] defined SQAVs, or "stable quasiabelian varieties" not intrinsically but as certain limits of abelian varieties. They are different from our varieties in several respects. In particular, some of Namikawa's varieties are not reduced, and it is not clear if they vary in flat families. The above definition also includes varieties which are not limits of abelian varieties.

The varieties of Definition 5.2 were called stable semiabelic varieties in [6]. Since this was a somewhat awkward name, I am reverting to the old name.

A stable quasiabelian pair is **linearized** if the sheaf $L = \mathcal{O}_X(B)$ is provided with a T-linearization (*not* G-linearization!) The first step is to classify the linearized varieties, which is quite easy:

Theorem 5.4. *A linearized stable quasiabelian variety $(G \curvearrowright X, L)$ is equivalent to the following data:*

(1) *(toric part) a linearized stable toric variety (X_0, L_0) for T, and*
(2) *(abelian part) a polarized abelian torsor (X_1, L_1) for A.*

The variety X is isomorphic to the twisted product

$$X = X_0 \times^T G = (X_0 \times G)/T, \quad T \text{ acting by } (t, t^{-1})$$

and there is a locally trivial fibration $X \to X_1$ with fibers isomorphic to X_0. Each closed orbit of $G \curvearrowright X$ with the restriction of L is isomorphic to the pair (X_1, L_1).

5.5. Hence, the linearized stable quasiabelian varieties (X, L) (resp. pairs (X, B)) over a field $k = \bar{k}$ are easy to classify, and are described by the following data:

(1) A complex of polytopes $\mathcal{Q}_0$ with a reference map $\tilde{\rho}_0 : |\mathcal{Q}_0| \to M_{0,\mathbb{R}}$.
(2) A polarized abelian torsor (X_1, L_1), which is equivalent to a polarized abelian variety $(A, \lambda : A \to A^t)$.
(3) A semiabelian variety G, which is equivalent to a homomorphism $c : M_0 \to A^t$.
(4) A certain cohomology group, very similar to the one in 4.3, describing the gluing. The only difference is that in this case the sheaves, instead of tori, have coefficients in certain $\mathbb{G}_m$-torsors.

The topological space $|\mathcal{Q}_0|$ has dimension $\dim X_0 \leq \dim T = r$, the toric rank of G. The analogy with stable toric varieties becomes even stronger when we associate to (X, L) a *cell* complex $\mathcal{Q}$ of dimension $\dim X$. This is done as follows:

The kernel of the polarization map $\lambda : A \to A^t$ has order d^2, where d is the degree of polarization, and comes with a skew-symmetric bilinear form. If $\mathrm{char}\, k \nmid d$, it can be written as $\ker \lambda = H \times \mathrm{Hom}(H, \mathbb{G}_m)$ for a unique finite abelian group H of rank $\leq \dim A$. Write H as a quotient M_1/Γ_1, where $M_1 = \mathbb{Z}^a$ and Γ_1 is a subgroup of finite index.

The cell complex $\mathcal{Q}_1$ we associate to the abelian torsor (X_1, L_1) consists of one cell $M_{1,\mathbb{R}}/\Gamma_1 \simeq \mathbb{R}^a/\mathbb{Z}^a$, a real torus of dimension a. Let $M = M_0 \times M_1$ and $\Gamma = \Gamma_1$. Then the cell complex associated to the linearized pair (X, L) is $\mathcal{Q} = \mathcal{Q}_0 \times \mathcal{Q}_1$. It comes with a reference map $\rho : |\mathcal{Q}| \to M_{\mathbb{R}}/\Gamma_1 \simeq \mathbb{R}^g/\mathbb{Z}^a$.

It turns out, however, that if $(T \curvearrowright X, B)$ is a degeneration of abelian varieties with $T \neq 1$, the invertible sheaf L is *never* linearized. On the other hand, there exists an infinite *algebraic* cover $f : \widetilde{X} \to X$ such that the pullback $\widetilde{L} = f^*L$ is linearized in a canonical way:

Theorem 5.6 ([6])**.** *Let G be a semiabelian group scheme over a connected scheme S, and assume that G is a global extension $1 \to T \to G \to A \to 0$ of an abelian scheme A by a split torus T. Then a family of stable quasiabelian pairs $(G \curvearrowright X, B) \to S$ is equivalent to a family of* linearized *stable quasiabelian pairs $(G \curvearrowright \widetilde{X}, \widetilde{B}) \to S$ whose fibers are only locally of finite type, with a compatible* free *in Zariski topology action of $M_0 = \mathbb{Z}^r$ so that $(X, B) = (\widetilde{X}, \widetilde{B})/M_0$.*

Moreover, there exists a subgroup $\Gamma_0 \simeq \mathbb{Z}^{r'}$, $r' \leq r$, of M_0 such that $\widetilde{X}$ is a disjoint union of $[M_0 : \Gamma_0]$ copies of a connected scheme $\widetilde{X}'$, and $(X, B) = (\widetilde{X}', \widetilde{B}')/\Gamma_0$.

A polarized toric variety (X, L) provides a trivial case of this theorem: in this case $\Gamma_0 = 0$, and X is the disjoint union of $M_0 \simeq \mathbb{Z}^r$ copies of X, one for each possible T-linearization of the sheaf L.

A less trivial, but equally familiar example is the rational nodal curve, which is a quotient $\widetilde{X}/\mathbb{Z}$ of an infinite chain of $\mathbb{P}^1$s by $M_0 \simeq \mathbb{Z}$. Mumford [42] constructed a number of degenerations of abelian varieties which are such infinite quotients. So the above statement may be considered to be a precise inverse of Mumford's construction.

The scheme $\widetilde{X}'$ can be written in the form $\widetilde{X}_0 \times^T G$, where $\widetilde{X}_0$ is a linearized scheme locally of finite type. Then locally $\widetilde{X}_0$ is isomorphic to a linearized stable toric variety. This defines a locally finite complex of polytopes $\widetilde{\mathcal{Q}}_0$ with a reference map $\tilde{\rho}_0 : |\widetilde{\mathcal{Q}}_0| \to M_{0,\mathbb{R}}$. Moreover, $\widetilde{\mathcal{Q}}_0$ has a Γ_0-action with only finitely many orbits, and $\tilde{\rho}_0$ is Γ_0-invariant. This can be summed up by saying that each stable quasiabelian pair defines a finite complex of polytopes $\rho_0 : \mathcal{Q}_0 = \widetilde{\mathcal{Q}}_0/\Gamma_0 \to M_{0,\mathbb{R}}/\Gamma_0 \simeq \mathbb{R}^r/\mathbb{Z}^{r'}$.

Again, we can add to this the abelian part, and obtain a complex of dimension $\dim X$. As above, the abelian part gives a one-cell complex $\mathcal{Q}_1 = M_{1,\mathbb{R}}/\Gamma_1 \simeq \mathbb{R}^a/\mathbb{Z}^a$. We set $\mathcal{Q} = \mathcal{Q}_0 \times \mathcal{Q}_1$, $M = M_0 \times M_1$, and $\Gamma = \Gamma_0 \times \Gamma_1$. Then $\mathcal{Q}$ is a finite cell complex, and it comes with a reference map $\rho : |\mathcal{Q}| \to M_{\mathbb{R}}/\Gamma \simeq \mathbb{R}^g/\mathbb{Z}^{a+r'}$.

The topological space $\rho : |\mathcal{Q}| \to M_{\mathbb{R}}/\Gamma$ together with the reference map is the **type of a stable quasiabelian variety** (X, L), resp. of a pair (X, B). We say

that the type is **injective** if the reference map ρ is injective. In this case it can be shown that the type is constant in connected families, and so we can talk about moduli spaces $M_{|\mathcal{Q}|}$.

5.7. The classification of isomorphism classes in the stable toric case 4.3 and linearized stable semiabelian case 5.5 can be translated to this most general case almost verbatim.

The most important of the moduli spaces $M_{|\mathcal{Q}|}$ are the ones containing abelian torsors of degree d, defining a polarization $\lambda : A \to A^t$ with $\ker \lambda \simeq H \times \operatorname{Hom}(H, \mathbb{G}_m)$. In this case, the type is a real torus $M_\mathbb{R}/\Gamma \simeq \mathbb{R}^g/\mathbb{Z}^g$, where $\Gamma \subset M$ is a sublattice with $M/\Gamma \simeq H$. One observes that this *real torus with a lattice structure* is an analogue of a lattice polytope Q in the stable toric case.

A cell subdivision of $|\mathcal{Q}|$ in this case is the same as a Γ-periodic subdivision of $M_\mathbb{R}$ which is a pullback of a Γ_0-periodic subdivision of $M_{0,\mathbb{R}}$ into lattice polytopes with vertices in M_0.

One proves that the moduli functor of stable quasiabelian pairs of injective type is proper, and that one-parameter degenerations correspond to suitably understood convex subdivisions of $|\mathcal{Q}|$.

A Γ-periodic subdivision of $M_\mathbb{R}$ is **convex** if it is the projection of the lower envelope of the points $(m, h(m))$, where m goes over $M \simeq \mathbb{Z}^g$, and $h : M \to \mathbb{R}$ is a function of the form

$$h(m) = (\text{positive semidefinite quadratic form}) + r(m \mod \Gamma),$$

where $r : M/\Gamma \to \mathbb{R}$ is a function defined on the finite set of residues.

In particular, the principally polarized case corresponds to $\Gamma = M = \mathbb{Z}^g$. The convex subdivisions of $\mathbb{R}^g/\mathbb{Z}^g$ in this case are the classical **Delaunay decompositions**, that appeared in [50]. A detailed combinatorial description of one-parameter degenerations of principally polarized abelian varieties from the present point of view, in which Delaunay decompositions naturally appear, was given in [15].

By analogy, when $\Gamma \subset M$ is a sublattice of finite index, we call the convex subdivisions of $M_\mathbb{R}/\Gamma$ **semi-Delaunay decompositions**.

We will denote the moduli spaces appearing in this case by $\overline{AP}_{g,H}$, resp. $\overline{AP}_g$ in the principally polarized case; AP stands for *abelian pairs*.

Theorem 5.8. *For each of the types $|\mathcal{Q}| = M_\mathbb{R}/\Gamma \simeq \mathbb{R}^g/\mathbb{Z}^g$, $|M/\Gamma| = d$, the functor of stable quasiabelian pairs of type $|\mathcal{Q}|$ over $\mathbb{Z}[1/d]$ has a coarse moduli space $\overline{AP}_{g,H}$, a proper algebraic space over $\mathbb{Z}[1/d]$. The moduli space $\overline{AP}_{g,H}$ has a natural stratification with strata corresponding to subdivisions of $|\mathcal{Q}|$ modulo symmetries of (M, Γ).*

The moduli space $\overline{AP}_{g,H}$ contains an open subset $U_{|\mathcal{Q}|}$ of dimension $\frac{g(g+1)}{2} + d - 1$ which is the moduli space of abelian torsors (X, B) defining a polarization of degree d. The closure of $U_{|\mathcal{Q}|}$ is an irreducible component of $\overline{AP}_{g,H}$. The strata in this closure correspond to semi-Delaunay subdivisions.

In particular, one has the following:

Theorem 5.9. *For $|\mathcal{Q}| = M_\mathbb{R}/M \simeq \mathbb{R}^g/\mathbb{Z}^g$, the functor of stable quasiabelian pairs of type $|\mathcal{Q}|$ has a coarse moduli space $\overline{AP}_g$, a proper algebraic space over $\mathbb{Z}$. The moduli space $\overline{AP}_g$ has a natural stratification with strata corresponding to $\mathbb{Z}^g$-periodic subdivisions of $\mathbb{R}^g$, pullbacks of $\mathbb{Z}^r$-periodic subdivisions of $\mathbb{R}^r$ into lattice polytopes with the set of vertices equal to the lattice $\mathbb{Z}^r$ of periods, modulo $\operatorname{GL}(g, \mathbb{Z})$.*

The moduli space $\overline{AP}_g$ contains an open subset which is the moduli space A_g of principally polarized abelian varieties. The closure of A_g is an irreducible component of $\overline{AP}_g$. The strata in this closure correspond to Delaunay subdivisions, and the normalization of $\overline{A}_g$ coincides with the toroidal compactification $\overline{A}_g^{\mathrm{vor}}$ for the second Voronoi fan. This toroidal compactification is projective.

The proof of the Theorems exploits the connection with the toric case in the following way. Although the toric rank in a family of stable quasiabelian varieties may change, in an infinitesimal family it does not. Hence, Theorem 5.6 together with the toric methods give the deformation and obstruction theory for the moduli functor. The moduli spaces then can be constructed by using Artin's method [16].

The (locally closed) cones of the second Voronoi fan consist of positive semidefinite quadratic forms that define the same Delaunay decomposition. Thus, we see that this fan, introduced by Voronoi in [50], is the precise infinite periodic analogue of (the normal fan of) the secondary polytope, and predates it by about 80 years.

Starting with $g = 4$, the moduli spaces $\overline{AP}_g$ do indeed have several irreducible components (as do the moduli spaces of stable toric varieties). The extra components always appear when there exists a non-Delaunay $\mathbb{Z}^r$-periodic subdivision of $\mathbb{R}^r$ into lattice polytopes with vertices in the same $\mathbb{Z}^r$, with $r \le g$.

Another situation where the extra components are guaranteed is when the stratum for a particular Delaunay decomposition has higher dimension in $\overline{AP}_g$ than it does in $\overline{A}_g^{\mathrm{vor}}$. Both can be computed effectively: for the Voronoi compactification it is the codimension of a cone in the 2nd Voronoi fan, and for $\overline{AP}_g$ it is the dimension of the cohomology group describing the gluing, as in 4.3, 5.5, 5.7. See [5] for more on this.

One important construction involving moduli spaces of abelian varieties is the Torelli map $M_g \to A_g$ which associates to a smooth curve C of genus g its Jacobian, a principally polarized abelian variety $(A, \lambda : A \to A^t)$ of dimension g. Combinatorially, it was understood by Mumford (see [43]) that the Torelli map can be extended to a morphism from the Deligne-Mumford compactification $\overline{M}_g$ to $\overline{A}_g^{\mathrm{vor}}$, and this was the original motivation for considering the second Voronoi fan in [43].

The moduli interpretation of the extended morphism $\overline{M}_g \to \overline{AP}_g$ was given in [7]. To a nonsingular curve C, it associates the pair $(\mathrm{Pic}^0 C \curvearrowright \mathrm{Pic}^{g-1} C, \Theta)$, and to a stable curve, the stable quasiabelian pair $(\mathrm{Pic}^0 C \curvearrowright \mathrm{Jac}_{g-1} C, \Theta)$, where $\mathrm{Jac}_{g-1} C$ is the moduli stable of semistable rank 1 sheaves on C of degree $g-1$, and Θ is the divisor corresponding to sheaves with sections.

The $\mathbb{Z}^g$-periodic cell decompositions corresponding to the image of $\overline{M}_g$ have a simple description. First of all, they are not arbitrary but are given by subdividing $\mathbb{R}^r$ (and by pullback, $\mathbb{R}^g$) by systems of parallel hyperplanes $\{l_i(m) = n \in \mathbb{Z}\}$. The condition that the set of vertices of the polytopes in $\mathbb{R}^r$ cut out by the hyperplanes is the same lattice of periods $\mathbb{Z}^r$ is equivalent to the condition that $\{l_i \in M_0^\vee\}$ is a **unimodular system of vectors**, i.e. every $(r \times r)$-minor is $0, 1$ or -1. Another name used for unimodular systems of vectors is **regular matroid**, see [45].

In these terms, the answer to "combinatorial Schottky" or "tropical Schottky" problem is the following: the strata in the image of $\overline{M}_g$ correspond to special matroids that are called *cographic*. If $\mathcal{G}$ is a graph then its cographic subdivision

is the $H_1(\mathcal{G}, \mathbb{Z})$-periodic subdivision of $H_1(\mathcal{G}, \mathbb{R})$ obtained by intersecting H_1 with $C_1(\mathcal{G}, \mathbb{R})$ divided into standard Euclidean cubes.

This theory was extended to the degenerations of Prym varieties in [8, 49], and many non-cographic matroids appear there. For example, using the above theory Gwena [21] described some of the degenerations of intermediate Jacobians of cubic 3-folds, including a particular one that corresponds to a very symmetric regular matroid R_{10} which is neither cographic, nor graphic.

In conclusion, we would like to mention one other important motivation from the Minimal Model Program for looking at stable abelian pairs: by a theorem of Kollár [37], if A is a principally polarized variety then the pair (A, Θ) has log canonical singularities.

6. Grassmannians

In Section 4, we defined stable toric (and spherical) varieties over a projective G-scheme $Y \subset \mathbb{P}^n$. The corresponding moduli spaces $M_{Y,Q}$ are projective. What does one get by looking at some particular varieties Y? One nice case that has many connections with other fields is the case when $Y \subset \mathbb{P}^n$ is a grassmannian with its Plücker embedding and the group is the multiplicative torus.

Let $E = E_1 \oplus \cdots \oplus E_n$ be a linear space with the coordinate-wise action by the torus $T = \mathbb{G}_m^n$ with character group $M = \mathbb{Z}^n$. (Dimensions of E_i are arbitrary.) Let r be a positive integer, and

$$i : Y = \mathrm{Gr}(r, E) \hookrightarrow \mathbb{P}(\Lambda^r E)$$

be the grassmannian variety of r-dimensional subspaces of E with its Plücker embedding.

For each collection of 2^n nonnegative integers $\underline{d} = (d_I \mid I \subset \{1, \ldots, n\})$, the **thin Schubert cell** is defined to be the locally closed subscheme of $\mathrm{Gr}(r, E)$

$$\mathrm{Gr}_{\underline{d}} = \{V \subset E \mid \mathrm{rank}(V \cap \oplus_{i \in I} E_i) = d_I\}$$

(Some of these may be empty; one necessary condition for non-emptiness is the inequality $d_{I \cap J} + d_{I \cup J} \geq d_I + d_J$ for all I, J.) There is a subtorus $T_{\underline{d}}$ acting trivially on $\mathrm{Gr}_{\underline{d}}$, and the quotient torus $(T/T_{\underline{d}})$ acts freely.

A **generalized matroid polytope** is a polytope in $\mathbb{R}^n$ defined by the inequalities

$$Q_{\underline{d}} = \{0 \leq x_i \leq \dim E_i, \sum_{i=1}^{n} x_i = r \text{ and } \sum_{i \in I} x_i \geq d_I \text{ for all } I\}$$

The classical **matroid polytopes** are a special case, when all $\dim E_i = 1$.

In [40], Lafforgue constructed certain compactifications of the quotients of thin Schubert cells

$$\mathrm{Gr}_{\underline{d}}/T = \mathrm{Gr}_{\underline{d}}/(T/T_{\underline{d}})$$

These compactifications have important applications in Langlands program, and they include compactifications of the homogeneous spaces $\mathrm{PGL}_r^{n-1}/\mathrm{PGL}_r$ equivariant with respect to the action by PGL_r (this is the case of all $\dim E_i = r$ and $d_I = 0$ for all $I \neq \{1, \ldots, n\}$).

As shown in [12], the main irreducible components of the moduli spaces $M_{\mathrm{Gr}(r,E),Q_{\underline{d}}}$ of stable toric varieties over $\mathrm{Gr}(r, E)$, as in Section 4, coincide with the Lafforgue's compactifications, at least up to normalization.

The connection is as follows. For each point $p \in \mathrm{Gr}_{\underline{d}}$, the normalization of the closure of the orbit $T \cdot p$ defines a T-toric variety $X \to \mathrm{Gr}(r, E)$ for the polytope $Q_{\underline{d}}$. This gives a canonical identification of $\mathrm{Gr}_{\underline{d}}/T$ with an open subset U of the moduli space $M_{\mathrm{Gr}(r,E),Q_{\underline{d}}}$. Since the latter is projective, this gives a moduli compactification of $\mathrm{Gr}_{\underline{d}}/T$.

A case of particular interest is when all $\dim E_i = 1$ and Q is the moment polytope of a generic point $p \in \mathrm{Gr}(r, n)$. In this case $Q_{\underline{d}} = \Delta(r, n)$, a *hypersimplex*. This case was considered by Kapranov in [29] who constructed a compactification he called the Chow quotient, by using the Chow variety.

The moduli space $M = M_{\mathrm{Gr}(r,n),\Delta(2,n)}$ in this case can be interpreted as a compactified moduli space of hyperplane arrangements. This interpretation, with a somewhat folk status, was recorded by Hacking-Keel-Tevelev in [23], along with many new results about this moduli space. We note that the latter paper uses the toric Hilbert scheme, rather than stable toric varieties over Y. But in this case the two points of view coincide, because matroid polytopes, as was observed in [23], are unimodular (by which we mean that the monoid of integral points in the cone over Q is generated by the integral points of Q). As a consequence, stable toric varieties over $\mathrm{Gr}(r, n)$ are actually T-invariant *subschemes* of $\mathrm{Gr}(r, n)$.

We now review this interpretation. Let $p \in \mathrm{Gr}^0(r, n)$ be a generic point, and $X = \overline{T \cdot p}$ be the orbit closure, isomorphic to a (normal) toric variety for the polytope $\Delta(r, n)$. Then $X \to \mathrm{Gr}(r, n)$ can be equivalently interpreted as any of the following:

    (1) a point in $\mathrm{Gr}^0(r, n)/T$,
    (2) a point of an open subset $U = U_{\mathrm{Gr}(r,n),\Delta(r,n)}$ of $M_{\mathrm{Gr}(r,n),\Delta(r,n)}$.
    (3) a point of an open subset U of the toric Hilbert scheme of $\mathrm{Gr}(r, n)$.

Now pick a generic point $e \in \mathbb{A}^n$ and consider the closed subvariety Y_e of X corresponding to the r-dimensional subspaces that contain e. This is Kapranov's *visible contour*. It is easy to see that:

    (1) For generic e, e' the varieties Y_e and $Y_{e'}$ are isomorphic since they differ by T-action. So we could as well associate one variety Y with each X, up to an isomorphism.
    (2) As the line $\mathbb{A}^1_e$ changes, the disjoint subvarieties Y_e cover an open subset V of X, so they are fibers of a proper fibration $V \to T/\mathbb{G}_m$. In particular, the singularities of $(Y_e, \Delta \cap Y_e)$ are no worse than singularities of (X, Δ), where Δ is the complement of the dense torus orbit in X.
    (3) In fact, each Y is isomorphic to $\mathbb{P}^{r-1}$, and $Y \cap \Delta = B_1 \cup B_2 \cup \cdots \cup B_n$ is the union of n hyperplanes in $\mathbb{P}^{r-1}$ in general position. The divisors B_i correspond to the n coordinate hyperplanes in $E = (\mathbb{A}^1)^n$.
    (4) By the Gelfand-MacPherson correspondence, the T-orbits of $\mathrm{Gr}^0(r, n)$ are in a natural bijection with PGL_r-orbits of n hyperplanes in $\mathbb{P}^{r-1}$ that are in general position, i.e. with isomorphism classes of labeled hyperplane arrangements $(\mathbb{P}^{r-1}, B_1 + \cdots + B_n)$. So, $U = U_{\mathrm{Gr}(r,n),\Delta(r,n)}$ is the moduli space of the general-position hyperplane arrangements.

Now look at any *stable* toric variety $X \to \mathrm{Gr}(r, n)$ of type $|\mathcal{Q}| = \Delta(r, n)$ and at a corresponding subvariety Y_e. Then the properties (1) and (2) above still hold. In particular, each Y is a generic section of X, and the singularities of $(Y, B_1 + \cdots + B_n = Y \cap \Delta)$ are no worse than the singularities of the pair (X, Δ). But by

Lemma 4.4 the latter are semi log canonical. This implies that each pair $(Y, B_1 + \cdots + B_n)$ is stable in the sense of Definition 1.1.

Question 6.1. Can the moduli spaces $M_{Y,Q}$ in the case when Y is a partial flag variety, for example a variety of two-step flags, be interpreted as the moduli space of stable maps?

We also note that [7] provides an interpretation of the morphism

$$M_{\mathrm{Gr}(r,n),\Delta(r,n)} \to M_{\mathbb{P}(\Lambda^e E),\Delta(r,n)}$$

as a toric analogue of the extended Torelli map $\overline{M}_g \to \overline{A}_g$.

7. HIGHER GROMOV-WITTEN THEORY

One of the exciting new frontiers for the moduli of stable pairs is the "higher-dimensional" GW theory, obtained by replacing the n-pointed stable curves $(X, B_1 + \ldots + B_n) \to Y$ by stable pairs with $\dim X \geq 2$. We list several questions in this direction.

Question 7.1. One way to define "higher" GW-invariants is to use evaluations at the intersections points $\cap_{j \in J} B_j$ with $|J| = \dim X$. Can a "generalized" quantum cohomology ring be defined using these evaluations? And is it a richer structure than simply an associative ring?

Question 7.2. Is there a more nontrivial definition, using evaluations at divisors rather than at points?

Question 7.3. Do the moduli spaces of weighted n-pointed curves, constructed by Hassett [27] lead to new ways to compute ordinary GW-invariants and descendants? When one varies the weights b_j and the moduli space $\overline{M}_{0,n}^{(b_j)}(\beta, Y)$ changes, is there a nice "wall-crossing" formula?

Question 7.4. The formula for the intersection products of ψ-classes on $\overline{M}_{0,n}$ is particularly easy (these are just the multinomial coefficients). What is the generalization of this formula for the compactified moduli space of hyperplane arrangements? Can it be obtained by using the toric degeneration of $\mathrm{Gr}(r,n)$ to a Gelfand-Tsetlin toric variety Z and thus degenerating $M_{\mathrm{Gr}(r,n),Q}$ to $M_{Z,Q}$?

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